Geometry of anti-Poisson involutions in the deformations and resolutions of Kleinian singularities

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7 - Lie Theory and Generalizations

1. Kleinian singularities
2. Anti-Poisson involutions and their fixed point loci
3. Preimages of fixed point loci under minimal resolutions
4. Deformations and connections to Lie theory

Overview:

$$\begin{tikzcd}[column sep=large]
\text{minimal resolution} & & \text{Kleinian singularity} & \\
\widetilde{X} \arrow[rr, "\pi"] & & X \arrow[loop right, "\theta"] & \text{anti-Poisson involution} \\
\cup & & \cup & \\
\pi^{-1}(X^\theta) & & X^\theta & \text{Lagrangian subvariety}
\end{tikzcd}$$

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as schemes.

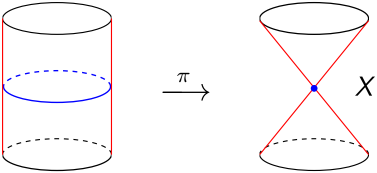
Motivation: Classify (certain) irreducible Harish-Chandra $(\mathfrak{g}, K)$-modules in geometric terms. Roughly speaking, Supp(HC modules) $\approx X^\theta$.

Extended Goal: Replace X by its Poisson deformation $X_\lambda,\ \lambda \in \mathfrak{h}/W$, and describe families of Lagrangian subvarieties X_λ^θ and $\pi^{-1}(X_\lambda^\theta)$.

An example

Type A_1 Kleinian singularity: $X = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy - z^2)$
Anti-Poisson involution $\theta : x \leftrightarrow y$

$\widetilde{X} = T^*\mathbb{P}^1$

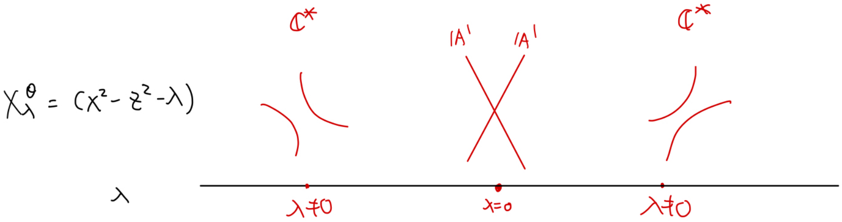


$X = \mathcal{N}$, nilpotent cone of $\mathfrak{sl}_2(\mathbb{C})$

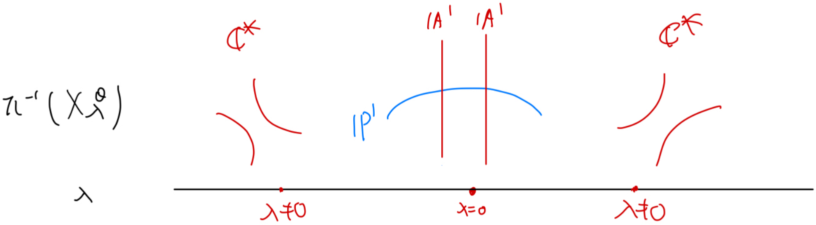
$X^\theta = \operatorname{Spec}\mathbb{C}[x,z]/(x^2 - z^2)$ is the union of two $\mathbb{A}^1$'s.
$\pi^{-1}(X^\theta)$ is the union of two $\mathbb{A}^1$'s and a $\mathbb{P}^1$.

Poisson deformation, $X_\lambda = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy - z^2 - \lambda) \curvearrowleft \theta$
$X_\lambda^\theta = \operatorname{Spec}\mathbb{C}[x,z]/(x^2 - z^2 - \lambda) \simeq \mathbb{C}^*$ if $\lambda \neq 0$.

An example



$X_\lambda^\theta = (x^2 - z^2 - \lambda)$
$\mathbb{C}^*$
$\mathbb{A}^1$
$\mathbb{A}^1$
$\mathbb{C}^*$
λ
$\lambda \neq 0$
$\lambda = 0$
$\lambda \neq 0$



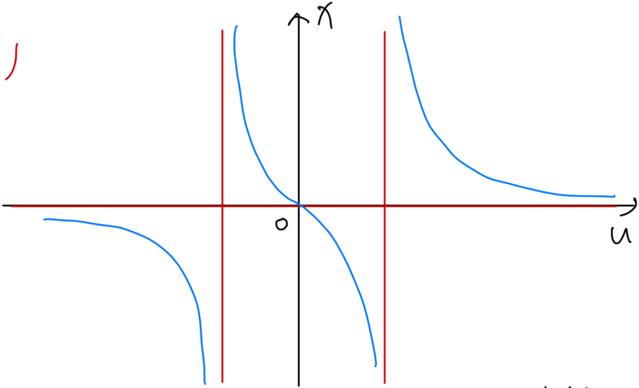
$\pi^{-1}(X_\lambda^\theta)$
$\mathbb{C}^*$
$\mathbb{A}^1$
$\mathbb{A}^1$
$\mathbb{C}^*$
$\mathbb{P}^1$
λ
$\lambda \neq 0$
$\lambda = 0$
$\lambda \neq 0$

An example

$\pi^{-1}(x_\lambda^\theta)$

$\lambda = 0$

$\lambda \neq 0$



$$x(u^2-1) = 2\lambda u \Rightarrow x = \frac{2\lambda u}{u^2-1}$$

$$z = xu - \lambda$$

Kleinian singularities

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup.
The **Kleinian singularities** is the quotient $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u, v]^\Gamma$.

Example

When $\Gamma = \{\pm I_2\}$, we have $\mathbb{C}[u, v]^\Gamma =$ even degree polynomials $=$ $\mathbb{C}[x = u^2, y = v^2, z = uv] = \mathbb{C}[x, y, z]/(xy - z^2)$.

Fact (Klein, 1884)

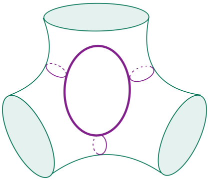
$\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (one relation), and $\mathbb{C}^2/\Gamma$ has an isolated singularity at 0.

Minimal resolutions

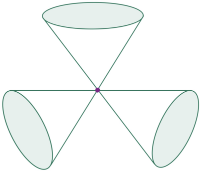
McKay correspondence: Kleinian singularities are in bijection with ADE Dynkin diagrams.

$\pi\colon \widetilde{X} \to X$, minimal resolution.
$\pi^{-1}(0) =$ union of $\mathbb{P}^1$'s, according dually to ADE Dynkin diagrams.

Example:



$\xrightarrow{\pi}$



D_4 singularity
$x^3 + xy^2 + z^2 = 0$

Anti-Poisson involutions & fixed point loci

Set $X := \mathbb{C}^2/\Gamma$. The algebra of functions $\mathbb{C}[X] = \mathbb{C}[u,v]^\Gamma$ is a graded Poisson algebra with **Poisson bracket**

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

Definition: An **anti-Poisson involution** of $X = \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\},\ \forall\ f_1, f_2 \in \mathbb{C}[X].$$

Definition: The **fixed point locus** is $X^\theta := \operatorname{Spec} \mathbb{C}[X]/I$, where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$.

Anti-Poisson involution & fixed-point loci

Proposition 1 (H.)

- There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$ up to conjugation by graded Poisson automorphisms.
- The fixed point locus X^θ is reduced.
- If X^θ is not a single point, each irreducible component of X^θ is either $\mathbb{A}^1$ or a cusp.

Example: Type A_n singularity $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1})$.
Then θ swapping $x \leftrightarrow y$ is an anti-Poisson involution. We have
$X^\theta = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1}, x - y) \simeq \operatorname{Spec} \mathbb{C}[x, z]/(x^2 - z^{n+1})$,
which is a union of two $\mathbb{A}^1$'s when n is odd, a cusp when n is even.

Examples in type A

Example: Anti-Poisson involutions for type A_n Kleinian singularities.

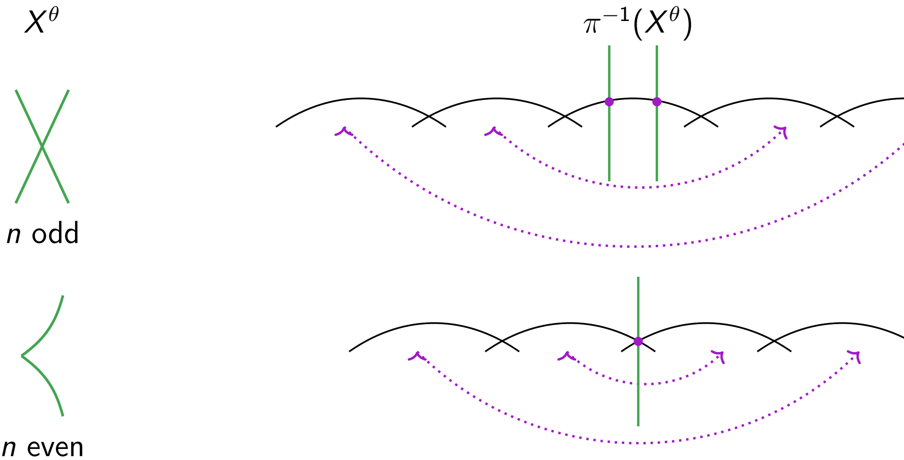
	Type I	Type II	Type III
A_n, n odd $xy - z^{n+1} = 0$	$x \mapsto y$ $y \mapsto x$ $z \mapsto z$	$x \mapsto x$ $y \mapsto y$ $z \mapsto -z$	$x \mapsto -x$ $y \mapsto -y$ $z \mapsto -z$
A_n, n even $xy - z^{n+1} = 0$	$x \mapsto y$ $y \mapsto x$ $z \mapsto z$	$x \mapsto -x$ $y \mapsto y$ $z \mapsto -z$	
involution of $\mathfrak{sl}_{n+1}$	$M \mapsto -M^t$	$\mathrm{Ad}(I_{\lceil \frac{n+1}{2} \rceil, \lfloor \frac{n+1}{2} \rfloor})$	$\mathrm{Ad}(I_{\frac{n+3}{2}, \frac{n-1}{2}})$

Fact: All anti-Poisson involutions come from Lie algebra involutions.

Preimage of fixed point loci under minimal resolutions

Recall $\pi\colon \widetilde{X} \to X$ minimal resolution. $0 \in X^\theta \Rightarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$.

Example: Consider type A_n singularities with θ swapping $x \leftrightarrow y$.



Straight green line is $\mathbb{A}^1$, curly black line is $\mathbb{P}^1$ Lift the actio

Lift anti-Poisson involutions

$\pi : \widetilde{X} \to X$, minimal resolution, and θ an anti-Poisson involution of X.

Theorem (H., 2025)

There exists a unique anti-symplectic involution $\tilde{\theta} : \widetilde{X} \to \widetilde{X}$ such that $\pi \circ \tilde{\theta} = \theta \circ \pi$. It can be constructed via involutions of quiver varieties.

We have $\pi^{-1}(X^\theta) = \pi^{-1}(0) \cup \widetilde{X}^{\tilde{\theta}}$.

Fact: $\widetilde{X}$ smooth $\Rightarrow$ $\widetilde{X}^{\tilde{\theta}}$ is smooth Lagrangian (no intersection, no cusp).

Remark: The preimage $\pi^{-1}(X^\theta)$ is **NOT** reduced in general.

Write $\pi^{-1}(X^\theta) = \sum_{j=1}^{m} 1 \cdot L_j + \sum_{i=1}^{n} a_i C_i$ as a divisor, with $L_j \simeq \mathbb{A}^1$, $C_i \simeq \mathbb{P}^1$. There are methods to determine the multiplicities a_i.

Slodowy slice: universal graded Poisson deformation

Let $\mathfrak{g}$ be a simple Lie algebra of types ADE and $\mathcal{N}$ its nilpotent cone. Take $e \in \mathbb{O}_{\text{subreg}}$ and consider the Slodowy slice $S = e + \mathfrak{z}_{\mathfrak{g}}(f)$.

Fact: The Slodowy slice S is the universal graded Poisson deformation of Kleinian singularity $S \cap \mathcal{N} \simeq X$.

Consider the morphism $\varphi\colon S \to \mathfrak{h}/W$ defined as the composition

$$\varphi\colon S \hookrightarrow \mathfrak{g} \to \mathfrak{g}/\!/G \simeq \mathfrak{h}/W.$$

Zero fiber: $\varphi^{-1}(0) = S \cap \mathcal{N} \simeq X$ is the Kleinian singularity.
Other fibers: $X_\lambda := \varphi^{-1}(\lambda)$ is a filtered Poisson deformation of X.

Question: Can we extend the action of θ to X_λ, for $\lambda \neq 0$?

Proposition 2 (H.)

θ naturally acts on S and $\mathfrak{h}/W$ by the universal property. The action of θ extends to X_λ iff $\lambda \in (\mathfrak{h}/W)^\theta$.

Lagrangians in deformations of Kleinian singularities

Question: How to describe X_λ^θ and $\pi^{-1}(X_\lambda^\theta)$, for $\lambda \neq 0$?

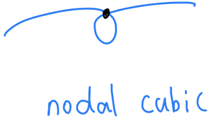
- Step 1: Determine the singular points of X_λ. Each is a Kleinian singularity of smaller type.
- Step 2: Identify the local Type of the anti-Poisson involution near each singular point.
- Step 3: Describe the local intersection pattern of projective lines and affine curves, and then connect these local pieces to obtain a global picture.

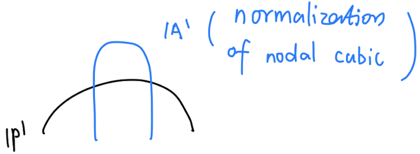
Examples

Example: An A_2 deformation with singularity type A_1

$$\mathcal{X}_\lambda = \operatorname{Spec} \mathbb{C}[x, y, z] / (xy - (z-1)^2(z+2)) \qquad \theta : x \leftrightarrow y$$

$$\mathcal{X}_\lambda^\theta = (x^2 - (z-1)^2(z+2)) \qquad \pi^{-1}(\mathcal{X}_\lambda^\theta)$$





Example: An A_3 deformation with singularity type $A_1 \times A_1$

$$\mathcal{X}_\lambda = \operatorname{Spec} \mathbb{C}[x, y, z] / (xy - (z-1)^2(z+1)^2) \qquad \theta : x \leftrightarrow y$$

Thank you!